On a nonlinear heat equation associated with Dirichlet – Robin conditions

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Abstract. *This paper is devoted to the study a nonlinear heat equation associated with Dirichlet-Robin conditions. At first, we use the Faedo – Galerkin and the compactness method to prove existence and uniqueness results. Next, we consider the properties of solutions. We obtain that if the initial condition is bounded then so is the solution and we also get asymptotic behavior of solutions as $t \to +\infty$. Finally, we give numerical results.*

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1 Introduction

In this paper, we consider the following nonlinear heat equation

$$u_t - \frac{\partial}{\partial x}\left[\mu\left(x,t\right)u_x\right] + f(u) = f_1(x,t),\ 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

associated with conditions

$$u_x(0,t) = h_0 u(0,t) + g_0(t),\ \ -u_x(1,t) = h_1 u(1,t) + g_1(t), \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

where u_0, μ, f, f_1, g_0, g_1 are given functions satisfying conditions, which will be specified later, and h_0, $h_1 \geq 0$ are given constants, with $h_0 + h_1 > 0$.

The conditions (1.2) are commonly known as Dirichlet – Robin conditions. They connect Dirichlet and Neumann conditions. Theses conditions arise from the effect of excess inert electrolytes in an electrochemical system through perturbation analysis ([2], [6], [7], [8]).

The governing equations (1.2) are the equation usually used in a diffusion, convection, migration transport system with electrochemical reactions occurring at the boundary electrodes and submitted to non linear constraints.

In electrochemistry, the oxidation-reduction reactions producing the current is modeled by a non linear elliptic boundary value problem, linearization of which gives the Dirichlet – Robin conditions ([3]). Theses conditions also appear in the response of an electrochemical thin film, such as separation in a micro – battery. His analyze is made by solving the Poisson – Nernst – Planck equation subject to boundary conditions appropriate (Dirichlet – Robin conditions) for an electrolytic cell ([4]).

The paper consists of six sections. In Section 2, we present some preliminaries. Using the Faedo – Galerkin method and the compactness method, in Section 3, we establish the existence of a unique weak solution of the problem (1.1) – (1.3) on $(0,T)$, for every $T > 0$. In section 4, we prove that if the initial condition is bounded, then so is the solution. In section 5, we study asymptotic behavior of the solution as $t \to +\infty$. In section 6 we give numerical results.

2 Preliminaries

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $|| \cdot ||$ stands for the norm in L^2 and we denote by $|| \cdot ||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \leq p \leq \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $||u||_{L^p(0,T;X)} < +\infty$, with

$$||u||_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p dt \right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ ess \sup\limits_{0<t<T} ||u(t)||_X, & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we shall use the following norms $\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{1/2}$, $\|v\|_i = \left(v^2(i) + \|v_x\|^2 \right)^{1/2}$, $i = 0, 1$.

Let $\mu \in C^0\left(\overline{Q}_T\right)$, with $\mu(x,t) \geq \mu_0 > 0$, for all $(x,t) \in \overline{Q}_T$, and the constants $h_0,\ h_1 \geq 0$, with $h_0 + h_1 > 0$, we consider a familly of symmetric bilinear forms $\{a(t;\cdot,\cdot)\}_{0\leq t\leq T}$ on $H^1 \times H^1$ as follows

$$\begin{aligned} a(t;u,v) &= \int_0^1 \mu(x,t)u_x(x)v_x(x)dx + h_0\mu\left(0,t\right)u(0)v(0) + h_1\mu\left(1,t\right)u(1)v(1) \\ &= \langle \mu(t)u_x, v_x\rangle + h_0\mu\left(0,t\right)u(0)v(0) + h_1\mu\left(1,t\right)u(1)v(1), \text{ for all } u,v \in H^1,\ 0 \leq t \leq T. \end{aligned} \tag{2.1}$$

Then we have the following lemmas.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\,\|v\|_{H^1}\,, \ \textit{for all } v \in H^1, \\ \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\,\|v\|_i\,, \ \textit{for all } v \in H^1,\ i = 0,1. \end{cases} \tag{2.2}$$

Lemma 2.2. *Let $\mu \in C^0\left(\overline{Q}_T\right)$, with $\mu(x,t) \geq \mu_0 > 0$, for all $(x,t) \in \overline{Q}_T$, and the constants $h_0,\ h_1 \geq 0$,* with *$h_0 + h_1 > 0$. Then, the symmetric bilinear form $a(t;\cdot,\cdot)$ is continuous on $H^1 \times H^1$ and coercive on H^1*, i.e.,

$$\begin{aligned} &(i) \qquad |a(t;u,v)| \leq a_T\,\|u\|_{H^1}\,\|v\|_{H^1}\,, \\ &(ii) \qquad a(t;v,v) \geq a_0\,\|v\|_{H^1}^2\,, \end{aligned} \tag{2.3}$$

for all $u,\ v \in H^1$, $0 \leq t \leq T$, where $a_T = (1 + 2h_0 + 2h_1)\sup\limits_{(x,t)\in\overline{Q}_T}\mu(x,t)$, and

$$a_0 = a_0(\mu_0, h_0, h_1) = \begin{cases} \mu_0\min\{h_0, \frac{1}{2}\}, & h_0 > 0,\ h_1 \geq 0, \\ \mu_0\min\{h_1, \frac{1}{2}\}, & h_1 > 0,\ h_0 \geq 0. \end{cases} \tag{2.4}$$

The proofs of these lemmas are straightforward. We shall omit the details.

Remark 2.1. It follows from (2.2) that on H^1, $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v\|_i$ are two equivalent norms satisfying

$$\tfrac{1}{\sqrt{3}}\,\|v\|_{H^1} \leq \|v\|_i \leq \sqrt{3}\,\|v\|_{H^1}\,, \text{ for all } v \in H^1,\ i = 0,1. \tag{2.5}$$

3 The existence and uniqueness theorem

We make the following assumptions:

(H_1) $h_0 \geq 0$ and $h_1 \geq 0$, with $h_0 + h_1 > 0$,

(H_2) $u_0 \in L^2$,

(H_3) $g_0,\, g_1 \in W^{1,1}(0,T)$,

(H_4) $\mu \in C^1([0,1] \times [0,T])$, $\mu(x,t) \geq \mu_0 > 0$, $\forall (x,t) \in [0,1] \times [0,T]$,

(H_5) $f_1 \in L^1(0,T;L^2)$,

(H_6) $f \in C^0(\mathbb{R})$ satisfies the condition, there exist positive constants $C_1,\ C_1',\ C_2$ and $p > 1$,

(i) $uf(u) \geq C_1 |u|^p - C_1'$,

$(2i)$ $|f(u)| \leq C_2(1 + |u|^{p-1})$, for all $u \in \mathbb{R}$.

The weak formulation of the initial boundary valued (1.1) – (1.3) can then be given in the following manner: Find $u(t)$ defined in the open set $(0,T)$ such that $u(t)$ satisfies the following variational problem

$$\frac{d}{dt}\langle u(t), v\rangle + a(t, u(t), v) + \langle f(u), v\rangle = \langle f_1(t), v\rangle - \mu\,(0,t)\,g_0(t)v(0) - \mu\,(1,t)\,g_1(t)v(1), \tag{3.1}$$

$\forall v \in H^1$, and the initial condition

$$u(0) = u_0. \tag{3.2}$$

We then have the following theorem.

Theorem 3.1. *Let $T > 0$ and $(H_1) - (H_6)$ hold. Then, there exists a weak solution u of problem* (1.1) – (1.3) *such that*

$$\begin{cases} u \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2), \\ tu \in L^\infty(0,T;H^1),\ tu_t \in L^2(0,T;L^2). \end{cases} \tag{3.3}$$

Furthermore, if f satisfies the following condition, in addition,

$$(H_7)\ \ (y - z)\,(f(y) - f(z)) \geq -\delta\,|y - z|^2\,,\ \textit{for all}\ y,\, z \in \mathbb{R},\ \textit{with}\ \delta > 0,$$

then the solution is unique.

Proof. The proof consists of several steps.

Step 1: *The Faedo – Galerkin approximation* (introduced by Lions [5]).

Let $\{w_j\}$ be a denumerable base of H^1. We find the approximate solution of the problem (1.1) – (1.3) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{3.4}$$

where the coefficients c_{mj} satisfy the system of linear differential equations

$$\begin{cases} \langle u_m'(t), w_j\rangle + a(t; u_m(t), w_j) + \langle f(u_m(t)), w_j\rangle \\ \qquad\qquad = \langle f_1(t), w_j\rangle - \mu(0,t)\, g_0(t) w_j(0) - \mu(1,t)\, g_1(t) w_j(1),\ 1 \le j \le m, \\ u_m(0) = u_{0m}, \end{cases} \tag{3.5}$$

where

$$u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \text{ strongly in } L^2. \tag{3.6}$$

It is clear that for each m there exists a solution $u_m(t)$ in form (3.4) which satisfies (3.5) and (3.6) almost everywhere on $0 \le t \le T_m$ for some T_m, $0 < T_m \le T$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. A priori estimates.

a) *The first estimate.* Multiplying the j^{th} equation of (3.5) by $c_{mj}(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} &\|u_m(t)\|^2 + 2\int_0^t a(s; u_m(s), u_m(s))ds + 2\int_0^t \langle f(u_m(s)), u_m(s)\rangle ds \\ &\quad = \|u_{0m}\|^2 + 2\int_0^t \langle f_1(s), u_m(s)\rangle ds \\ &\quad -2\int_0^t \mu(0,s)\, g_0(s) u_m(0,s) ds - 2\int_0^t \mu(1,s)\, g_1(s) u_m(1,s) ds. \end{aligned} \tag{3.7}$$

By $u_{0m} \to u_0$ strongly in L^2, we have

$$\|u_{0m}\|^2 \le C_0, \text{ for all } m, \tag{3.8}$$

where C_0 always indicates a bound depending on u_0.

By the assumptions $(H_6, (i))$, and using the inequalities (2.2), (2.3), and with $\beta > 0$, we estimate without difficulty the following terms in (3.7) as follows

$$2\int_0^t a(s; u_m(s), u_m(s))ds \ge 2a_0 \int_0^t \|u_m(s)\|_{H^1}^2 ds, \tag{3.9}$$

$$2\int_0^t \langle f(u_m(s)), u_m(s)\rangle ds \ge 2C_1 \int_0^t \|u_m(s)\|_{L^p}^p ds - 2TC_1', \tag{3.10}$$

$$2\int_0^t \langle f_1(s), u_m(s)\rangle ds \le \|f_1\|_{L^1(0,T;L^2)} + \int_0^t \|f_1(s)\| \|u_m(s)\|^2 ds, \tag{3.11}$$

$$\begin{aligned} -2\int_0^t \mu(0,s)\, g_0(s) u_m(0,s) ds &\le 2\sqrt{2}\, \|\mu\|_{L^\infty(Q_T)} \|g_0\|_{L^\infty} \int_0^t \|u_m(s)\|_{H^1} ds \\ &\le \tfrac{2}{\beta} T \|\mu\|_{L^\infty(Q_T)}^2 \|g_0\|_{L^\infty}^2 + \beta \int_0^t \|u_m(s)\|_{H^1}^2 ds, \end{aligned} \tag{3.12}$$

$$\begin{aligned} -2\int_0^t \mu(1,s)\, g_1(s) u_m(1,s) ds &\le 2\sqrt{2}\, \|\mu\|_{L^\infty(Q_T)} \|g_1\|_{L^\infty} \int_0^t \|u_m(s)\|_{H^1} ds \\ &\le \tfrac{2}{\beta} T \|\mu\|_{L^\infty(Q_T)}^2 \|g_1\|_{L^\infty}^2 + \beta \int_0^t \|u_m(s)\|_{H^1}^2 ds, \end{aligned} \tag{3.13}$$

for all $\beta > 0$. Hence, it follows from (3.7) – (3.13) that

$$\begin{aligned} &\|u_m(t)\|^2 + 2(a_0-\beta)\int_0^t \|u_m(s)\|_{H^1}^2\,ds + 2C_1\int_0^t \|u_m(s)\|_{L^p}^p\,ds \\ &\qquad \le C_0 + 2TC_1' + \|f_1\|_{L^1(0,T;L^2)} + \int_0^t \|f_1(s)\|\,\|u_m(s)\|^2\,ds \\ &\qquad\quad + \tfrac{2}{\beta}T\|\mu\|_{L^\infty(Q_T)}^2\left(\|g_0\|_{L^\infty}^2 + \|g_1\|_{L^\infty}^2\right). \end{aligned} \tag{3.14}$$

Choosing $\beta = \frac{1}{2}a_0$, we deduce from (3.14), that

$$S_m(t) \le C_T^{(1)} + \int_0^t C_T^{(2)}(s)S_m(s)ds, \tag{3.15}$$

where

$$\begin{cases} S_m(t) = \|u_m(t)\|^2 + a_0\int_0^t \|u_m(s)\|_{H^1}^2\,ds + 2C_1\int_0^t \|u_m(s)\|_{L^p}^p\,ds, \\ C_T^{(1)} = C_0 + 2TC_1' + \|f_1\|_{L^1(0,T;L^2)} + \frac{4}{a_0}T\|\mu\|_{L^\infty(Q_T)}^2\left(\|g_0\|_{L^\infty}^2 + \|g_1\|_{L^\infty}^2\right), \\ C_T^{(2)}(s) = \|f_1(s)\|\,,\ \ C_T^{(2)} \in L^1(0,T). \end{cases} \tag{3.16}$$

By the Gronwall's lemma, we obtain from (3.15), that

$$S_m(t) \le C_T^{(1)}\exp\left(\int_0^t C_T^{(2)}(s)ds\right) \le C_T, \tag{3.17}$$

for all $m \in \mathbb{N}$, for all t, $0 \le t \le T_m \le T$, i.e., $T_m = T$, where C_T always indicates a bound depending on T.

b) *The second estimate.* Multiplying the j^{th} equation of the system (3.5) by $t^2c_{mj}'(t)$ and summing up with respect to j, we have

$$\begin{aligned} &\|tu_m'(t)\|^2 + t^2a(t;u_m(t),u_m'(t)) + \langle tf(u_m(t)), tu_m'(t)\rangle \\ &\qquad = \langle tf_1(t), tu_m'(t)\rangle - t^2\mu(0,t)\,g_0(t)u_m'(0,t) - t^2\mu(1,t)\,g_1(t)u_m'(1,t). \end{aligned} \tag{3.18}$$

First, we need the following lemmas.

Lemma 3.2.

(i) $\frac{\partial a}{\partial t}(t;u,v) = \langle \mu'(\cdot,t)\,u_x, v_x\rangle + h_0\mu'(0,t)\,u(0)v(0) + h_1\mu'(1,t)\,u(1)v(1)$, *for all* $u,v \in H^1$,

(ii) $\left|\frac{\partial a}{\partial t}(t;u,v)\right| \le \widetilde{a}_T\|u\|_{H^1}\|v\|_{H^1}$, *for all* $u,v \in H^1$,

(iii) $\frac{d}{dt}a(t;u_m(t),u_m(t)) = 2a(t;u_m(t),u_m'(t)) + \frac{\partial a}{\partial t}(t;u_m(t),u_m(t))$, (3.19)

where $\widetilde{a}_T = (1+2h_0+2h_1)\sup\limits_{(x,t)\in[0,1]\times[0,T]}\mu'(x,t)$.

Lemma 3.3. *Put* $\lambda_0 = \left(\frac{C_1'}{C_1}\right)^{1/p}$, $m_0 = \int_{-\lambda_0}^{\lambda_0}|f(y)|\,dy$, and $\overline{f}(z) = \int_0^z f(y)dy$, $z \in \mathbb{R}$.

Then we have

$$-m_0 \leq \overline{f}(z) \leq C_2(|z| + \tfrac{1}{p}\,|z|^p), \ \ \forall z \in \mathbb{R}. \tag{3.20}$$

The proofs of these lemmas are straightforward. We shall omit the details.■
By $(3.19)_3$, we rewrite (3.18) as follows

$$\begin{aligned} &2\,\|tu_m'(t)\|^2 + \tfrac{d}{dt}a(t;tu_m(t),tu_m(t)) + 2\langle tf(u_m(t)),tu_m'(t)\rangle \\ &\qquad = 2ta(t;u_m(t),u_m(t)) + \tfrac{\partial a}{\partial t}(t;tu_m(t),tu_m(t)) + 2\langle tf_1(t),tu_m'(t)\rangle \\ &\qquad -2t^2\mu\,(0,t)\,g_0(t)u_m'(0,t) - 2t^2\mu\,(1,t)\,g_1(t)u_m'(1,t). \end{aligned} \tag{3.21}$$

Integrating (3.21), we get

$$\begin{aligned} &2\int_0^t \|su_m'(s)\|^2\,ds + a(t;tu_m(t),tu_m(t)) + 2\int_0^t\langle sf(u_m(s)),su_m'(s)\rangle ds \\ &\qquad = 2\int_0^t sa(s;u_m(s),u_m(s))ds + \int_0^t \tfrac{\partial a}{\partial t}(s;su_m(s),su_m(s))ds + 2\int_0^t\langle sf_1(s),su_m'(s)\rangle ds \\ &\qquad -2\int_0^t s^2\mu\,(0,s)\,g_0(s)u_m'(0,s)ds - 2\int_0^t s^2\mu\,(1,s)\,g_1(s)u_m'(1,s)ds. \end{aligned} \tag{3.22}$$

We shall estimate the terms of (3.22) as follows.

$$a(t;tu_m(t),tu_m(t)) \geq a_0\,\|tu_m(t)\|_{H^1}^2\,, \tag{3.23}$$

$$\begin{aligned} 2\int_0^t\langle sf(u_m(s)),su_m'(s)\rangle ds &= 2\int_0^t s^2ds\tfrac{d}{ds}\int_0^1 dx\int_0^{u_m(x,s)} f(y)dy \\ &= 2\int_0^t s^2ds\tfrac{d}{ds}\int_0^1 \overline{f}(u_m(x,s))dx \\ &= 2\int_0^t \left[\tfrac{d}{ds}\left(s^2\int_0^1 \overline{f}(u_m(x,s))dx\right) - 2s\int_0^1 \overline{f}(u_m(x,s))dx\right]ds \\ &= 2t^2\int_0^1 \overline{f}(u_m(x,t))dx - 4\int_0^t sds\int_0^1 \overline{f}(u_m(x,s))dx \\ &\geq -2T^2m_0 - 4C_2\int_0^t s\left[\|u_m(s)\|_{L^1} + \tfrac{1}{p}\,\|u_m(s)\|_{L^p}^p\ \right]ds \\ &\geq -2T^2m_0 - 4TC_2\left[T\,\|u_m\|_{L^\infty(0,T;L^2)} + \tfrac{1}{p}\tfrac{1}{2C_1}S_m(t)\right] \geq -C_T, \end{aligned} \tag{3.24}$$

$$2\int_0^t sa(s;u_m(s),u_m(s))ds \leq 2Ta_T\int_0^t \|u_m(s)\|_{H^1}^2\,ds \leq 2Ta_T\tfrac{1}{a_0}S_m(t) \leq C_T, \tag{3.25}$$

$$\begin{aligned} \int_0^t \tfrac{\partial a}{\partial t}(s;su_m(s),su_m(s))ds &\leq \widetilde{a}_T\int_0^t \|su_m(s)\|_{H^1}^2\,ds \leq T^2\widetilde{a}_T\int_0^t \|u_m(s)\|_{H^1}^2\,ds \\ &\leq T^2\widetilde{a}_T\tfrac{1}{a_0}S_m(t) \leq C_T, \end{aligned} \tag{3.26}$$

$$2\int_0^t\langle sf_1(s), su_m'(s)\rangle ds \le 2\int_0^t \|sf_1(s)\|\,\|su_m'(s)\|\,ds \le \int_0^t \|sf_1(s)\|^2\,ds + \int_0^t \|su_m'(s)\|^2\,ds$$
$$\le T^2\int_0^T \|f_1(s)\|^2\,ds + \int_0^t \|su_m'(s)\|^2\,ds$$
$$\le C_T + \int_0^t \|su_m'(s)\|^2\,ds. \tag{3.27}$$

By using integration by parts, it follows that

$$\left|-2\int_0^t s^2\mu(0,s)\,g_0(s)u_m'(0,s)ds\right|$$
$$= \left|-2t^2\mu(0,t)\,g_0(t)u_m(0,t) + 2\int_0^t \left[s^2\mu(0,s)\,g_0(s)\right]' u_m(0,s)ds\right|$$
$$\le 2\sqrt{2}t^2\|\mu\|_{L^\infty(Q_T)}\|g_0\|_{L^\infty}\|u_m(t)\|_{H^1} + 2\sqrt{2}\int_0^t \left|\left[s^2\mu(0,s)\,g_0(s)\right]'\right|\|u_m(s)\|_{H^1}\,ds$$
$$\le \tfrac{2}{\beta}T^2\|\mu\|^2_{L^\infty(Q_T)}\|g_0\|^2_{L^\infty} + \beta\|tu_m(t)\|^2_{H^1} + 2\sqrt{2}\int_0^t \left|\left[s^2\mu(0,s)\,g_0(s)\right]'\right|\|u_m(s)\|_{H^1}\,ds$$
$$\le \tfrac{1}{\beta}C_T + \beta\|tu_m(t)\|^2_{H^1} + 2\sqrt{2}\int_0^t \left|\left[s^2\mu(0,s)\,g_0(s)\right]'\right|\|u_m(s)\|_{H^1}\,ds. \tag{3.28}$$

On the other hand

$$\left|\left[s^2\mu(0,s)\,g_0(s)\right]'\right| = \left|2s\mu(0,s)\,g_0(s) + s^2\left[\mu'(0,s)\,g_0(s) + \mu(0,s)\,g_0'(s)\right]\right|$$
$$\le 2s\|\mu\|_{L^\infty(Q_T)}\|g_0\|_{L^\infty} + s^2\|\mu\|_{C^1(\overline{Q}_T)}\left[\|g_0\|_{L^\infty} + |g_0'(s)|\right] \tag{3.29}$$
$$\le s\|\mu\|_{C^1(\overline{Q}_T)}\left[(2+T)\|g_0\|_{L^\infty} + T|g_0'(s)|\right] \le sC_T\psi_0(s),$$

where

$$C_T = \|\mu\|_{C^1(\overline{Q}_T)}\left[(2+T)\|g_0\|_{L^\infty} + T\,\right],\ \ \psi_0(s) = 1 + |g_0'(s)|,\ \ \psi_0 \in L^1(0,T). \tag{3.30}$$

Hence, we deduce from (3.28), (3.29), that

$$\left|-2\int_0^t s^2\mu(0,s)\,g_0(s)u_m'(0,s)ds\right| \le \tfrac{1}{\beta}C_T + \beta\|tu_m(t)\|^2_{H^1} + 2\sqrt{2}C_T\int_0^t \psi_0(s)\|su_m(s)\|_{H^1}\,ds$$
$$\le \tfrac{1}{\beta}C_T + \beta\|tu_m(t)\|^2_{H^1} + 2C_T^2\int_0^T \psi_0(s)ds + \int_0^t \psi_0(s)\|su_m(s)\|^2_{H^1}\,ds$$
$$\le (1+\tfrac{1}{\beta})C_T + \beta\|tu_m(t)\|^2_{H^1} + \int_0^t \psi_0(s)\|su_m(s)\|^2_{H^1}\,ds, \tag{3.31}$$

for all $\beta > 0$.

Similarly

$$-2\int_0^t s^2\mu(1,s)\,g_1(s)u_m'(1,s)ds \le (1+\tfrac{1}{\beta})C_T + \beta\|tu_m(t)\|^2_{H^1} + \int_0^t \psi_1(s)\|su_m(s)\|^2_{H^1}\,ds, \tag{3.32}$$

for all $\beta > 0$, where

$$C_T = \|\mu\|_{C^1(\overline{Q}_T)}\left[(2+T)\|g_1\|_{L^\infty} + T\,\right],\ \ \psi_1(s) = 1 + |g_1'(s)|,\ \ \psi_1 \in L^1(0,T). \tag{3.33}$$

It follows from (3.22) – (3.27), (3.31) and (3.32), that

$$\begin{aligned}
&\int_0^t \|su_m'(s)\|^2\, ds + a_0 \|tu_m(t)\|_{H^1}^2 \\
&\qquad \leq (6+\tfrac{2}{\beta})C_T + 2\beta \|tu_m(t)\|_{H^1}^2 + \int_0^t \psi_0(s) \|su_m(s)\|_{H^1}^2\, ds \\
&\qquad + \int_0^t \psi_1(s) \|su_m(s)\|_{H^1}^2\, ds.
\end{aligned} \tag{3.34}$$

Choosing $2\beta = \frac{1}{2}a_0$, we deduce from (3.34), that

$$X_m(t) \leq \overline{C}_T^{(1)} + \int_0^t \overline{C}_T^{(2)}(s) X_m(s) ds, \tag{3.35}$$

where

$$\begin{cases}
X_m(t) = \|tu_m(t)\|_{H^1}^2 + \int_0^t \|su_m'(s)\|^2\, ds, \\
\overline{C}_T^{(1)} = \left(1+\frac{2}{a_0}\right)(6+\frac{8}{a_0})C_T, \\
\overline{C}_T^{(2)}(s) = \left(1+\frac{2}{a_0}\right)(\psi_0(s)+\psi_1(s)), \quad \overline{C}_T^{(2)} \in L^1(0,T).
\end{cases} \tag{3.36}$$

By the Gronwall's lemma, we obtain from (3.35), that

$$\|tu_m(t)\|_{H^1}^2 + \int_0^t \|su_m'(s)\|^2\, ds \leq \overline{C}_T^{(1)} \exp\left(\int_0^T \overline{C}_T^{(2)}(s) ds\right) \leq C_T, \tag{3.37}$$

for all $m \in \mathbb{N}$, for all $t \in [0,T]$, $\forall T > 0$, where C_T always indicates a bound depending on T.

Step 3. The limiting process.

By (3.16), (3.17) and (3.37) we deduce that, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$\begin{cases}
u_m \to u & \text{in} \quad L^\infty(0,T;L^2) \quad \text{weak*}, \\
u_m \to u & \text{in} \quad L^2(0,T;H^1) \quad \text{weak}, \\
tu_m \to tu & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weak*}, \\
(tu_m)' \to (tu)' & \text{in} \quad L^2(Q_T) \quad \text{weak}, \\
u_m \to u & \text{in} \quad L^p(Q_T) \quad \text{weak}.
\end{cases} \tag{3.38}$$

Using a compactness lemma ([5], Lions, p. 57) applied to $(3.38)_{3,4}$, we can extract from the sequence $\{u_m\}$ a subsequence still denotes by $\{u_m\}$, such that

$$tu_m \to tu \quad \text{strongly in } L^2(Q_T). \tag{3.39}$$

By the Riesz- Fischer theorem, we can extract from $\{u_m\}$ a subsequence still denoted by $\{u_m\}$, such that

$$u_m(x,t) \to u(x,t) \quad \text{a.e.} \quad (x,t) \quad \text{in} \quad Q_T = (0,1)\times(0,T). \tag{3.40}$$

Because f is continuous, then

$$f(u_m(x,t)) \to f(u(x,t)) \;\; \text{a.e.} \;\; (x,t) \;\; \text{in} \;\; Q_T = (0,1) \times (0,T). \tag{3.41}$$

On the other hand, by (H_6, ii), it follows from (3.16), (3.17) that

$$\|f(u_m)\|_{L^{p'}(Q_T)} \leq C_T, \tag{3.42}$$

where C_T is a constant independent of m.

We shall now require the following lemma, the proof of which can be found in [5].

Lemma 3.4. *Let Q be a bounded open set of $\mathbb{R}^N$ and $G_m, G \in L^q(Q)$, $1 < q < \infty$, such that,*

$$\|G_m\|_{L^q(Q)} \leq C, \;\; \text{where } C \text{ is a constant independent of } m,$$

and

$$G_m \to G \;\; \text{a.e.} \;\; (x,t) \;\; \text{in} \;\; Q.$$

Then

$$G_m \to G \;\; \text{in} \;\; L^q(Q) \;\; \text{weakly.} \blacksquare$$

Applying Lemma 3.4 with $N = 2$, $q = p'$, $G_m = f(u_m)$, $G = f(u)$, we deduce from (3.41), (3.42), that

$$f(u_m) \to f(u) \;\; \text{in} \;\; L^{p'}(Q_T) \;\; \text{weakly.} \tag{3.43}$$

Passing to the limit in (3.5) by (3.6), (3.38), (3.43), we have satisfying the equation

$$\begin{cases} \frac{d}{dt}\langle u(t), v\rangle + a(t, u(t), v) + \langle f(u), v\rangle \\ \qquad\qquad = \langle f_1(t), v\rangle - \mu\,(0,t)\,g_0(t)v(0) - \mu\,(1,t)\,g_1(t)v(1), \; \forall v \in H^1, \\ u(0) = u_0. \end{cases} \tag{3.44}$$

Step 4. Uniqueness of the solutions.

First, we shall need the following Lemma.

Lemma 3.5. *Let u be the weak solution of the following problem*

$$\begin{cases} u_t - \frac{\partial}{\partial x}\left[\mu\,(x,t)\,u_x\right] = \widetilde{f}(x,t), \; 0 < x < 1, \; 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = 0, \\ u \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2) \cap L^p(Q_T), \\ tu \in L^\infty(0,T;H^1), \;\; tu_t \in L^2(Q_T). \end{cases} \tag{3.45}$$

Then

$$\|u(t)\|^2 + 2\int_0^t a(s, u(s), u(s))ds = 2\int_0^t \langle \widetilde{f}(s), u(s)\rangle ds. \tag{3.46}$$

The lemma 3.5 is a slight improvement of a lemma used in [1] (see also Lions's book [5]).$\blacksquare$

Now, we will prove the uniqueness of the solutions. Assume now that (H_7) is satisfied.

Let u_1 and u_2 be two weak solutions of (1.1) – (1.3). Then $u = u_1 - u_2$ is a weak solution of the following problem (3.45) with the right hand side function replaced by $\widetilde{f}(x,t) = -f(u_1) + f(u_2)$. Using Lemma 3.5 we have equality

$$\|u(t)\|^2 + 2\int_0^t a(s, u(s), u(s))ds = -2\int_0^t \langle f(u_1) - f(u_2), u(s)\rangle ds. \tag{3.47}$$

Using the monotonicity of $f(y) + \delta y$, we obtain

$$\int_0^t \langle f(u_1) - f(u_2), u(s)\rangle ds \geq -\delta \int_0^t \|u(s)\|^2 ds. \tag{3.48}$$

It follows from (3.47), (3.48) that

$$\|u(t)\|^2 + 2a_0 \int_0^t \|u(s)\|_{H^1}^2 ds \leq 2\delta \int_0^t \|u(s)\|^2 ds. \tag{3.49}$$

By the Gronwall's Lemma that $u = 0$.

Therefore, Theorem 3.1 is proved.■

4 The boundedness of the solution

We now turn to the boundness of the solutions. For this purpose, we shall make of the following assumptions

(H_1') $h_0 > 0$ and $h_1 > 0$,

(H_2') $u_0 \in L^\infty$,

(H_5') $f_1 \in L^2(Q_T)$, $f_1(x,t) \leq 0$, *a.e.* $(x,t) \in Q_T$,

(H_6') $f \in C^0(\mathbb{R})$ satisfies the assumptions (H_6), (H_7), and

$$uf(u) \geq 0, \quad \forall u \in \mathbb{R}, \quad |u| \geq \|u_0\|_{L^\infty}.$$

We then have the following theorem.

Theorem 4.1. *Let* (H_1'), (H_2'), (H_3), (H_4), (H_5'), (H_6') *hold. Then the unique weak solution of the initial and boundary value problem* (1.1) – (1.3), *as given by theorem* 3.1*, belongs to* $L^\infty(Q_T)$.

Furthermore, we have also

$$\|u\|_{L^\infty(Q_T)} \leq \max\left\{\|u_0\|_{L^\infty}, \ \frac{1}{h_0}\|g_0\|_{L^\infty(0,T)}, \frac{1}{h_1}\|g_1\|_{L^\infty(0,T)}\right\}.■ \tag{4.1}$$

Remark 4.1. Assumption (H_2') is both physically and mathematically natural in the study of partial differential equation of the kind of (1.1) – (1.3), by means of the maximum principle.

Proof of Theorem 4.1. First, let us assume that

$$u_0(x) \leq M, \ a.e., \ x \in \Omega, \ \text{ and } \ \max\left\{\frac{1}{h_0}\|g_0\|_{L^\infty(0,T)}, \frac{1}{h_1}\|g_1\|_{L^\infty(0,T)}\right\} \leq M. \tag{4.2}$$

Then $z = u - M$ satisfies the initial and boundary value

$$\begin{cases} z_t - \frac{\partial}{\partial x}\left[\mu\left(x,t\right) z_x\right] + f(z+M) = f_1(x,t), \ 0<x<1, \ 0<t<T, \\ z_x(0,t) = h_0\left[z(0,t)+M\right] + g_0(t), \quad -z_x(1,t) = h_1\left[z(1,t)+M\right] + g_1(t), \\ z(x,0) = u_0(x) - M. \end{cases} \tag{4.3}$$

Multiplying equation $(4.3)_1$ by v, for $v \in H^1$ integrating by parts with respect to variable x and taking into account boundary condition $(4.3)_2$, one has after some rearrangements

$$\begin{aligned} &\int_0^1 z_t v dx + \int_0^1 \mu\left(x,t\right) z_x v_x dx + \mu\left(0,t\right)\left[h_0(z(0,t)+M) + g_0(t)\ \right] v(0) \\ &\qquad + \mu\left(1,t\right)\left[h_1(z(1,t)+M) + g_1(t)\ \right] v(1) \\ &\qquad + \int_0^1 f(z+M) v dx = \int_0^1 f_1(x,t) v dx, \quad \text{for all} \quad v \in H^1. \end{aligned} \tag{4.4}$$

Noticing from assumption (H_1') we deduce that the solution of the initial and boundary value problem (1.1) – (1.3) belongs to $L^2(0,T;H^1) \cap L^\infty(0,T;L^2) \cap L^p(Q_T)$, so that we are allowed to take $v = z^+ = \frac{1}{2}(|z|+z)$ in (4.4). Thus, it follows that

$$\begin{aligned} &\int_0^1 z_t z^+ dx + \int_0^1 \mu\left(x,t\right) z_x z_x^+ dx + \mu\left(0,t\right)\left[h_0(z(0,t)+M) + g_0(t)\ \right] z^+(0,t) \\ &\qquad + \mu\left(1,t\right)\left[h_1(z(1,t)+M) + g_1(t)\ \right] z^+(1,t) \\ &\qquad + \int_0^1 f(z+M) z^+ dx = \int_0^1 f_1(x,t) z^+ dx. \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|z^+(t)\right\|^2 + a(t, z^+(t), z^+(t)) + \int_0^1 f(z^+ + M) z^+ dx = \int_0^1 f_1(x,t) z^+ dx \\ &\qquad - \mu\left(0,t\right)\left(h_0 M + g_0(t)\right) z^+(0,t) - \mu\left(1,t\right)\left(h_1 M + g_1(t)\right) z^+(1,t) \le 0. \end{aligned} \tag{4.6}$$

since

$M \ge \max\{\frac{1}{h_0}\left\|g_0\right\|_{L^\infty}, \frac{1}{h_1}\left\|g_1\right\|_{L^\infty}\}$ and

$$\int_0^1 z_t z^+ dx = \int_{0,\ z>0}^1 (z^+)_t z^+ dx = \frac{1}{2}\frac{d}{dt}\int_{0,\ z>0}^1 \left|z^+\right|^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \left|z^+\right|^2 dx = \frac{1}{2}\frac{d}{dt}\left\|z^+(t)\right\|^2. \tag{4.7}$$

and on the domain $z > 0$ we have $z^+ = z$ and $z_x = (z^+)_x$.

On the other hand, by the assumption (H_2') and the inequality (2.3), we obtain

$$a(t, z^+(t), z^+(t)) \ge a_0 \left\|z^+(t)\right\|_{H^1}^2. \tag{4.8}$$

Using the monotonicity of $f(z) + \delta z$ and (H_7) we obtain

$$\begin{aligned} &\int_0^1 f(z^+ + M) z^+ dx = \int_0^1 \left[f(z^+ + M) - f(M)\right] z^+ dx + \int_0^1 f(M) z^+ dx \\ &\qquad \ge -\delta\int_0^1 \left|z^+\right|^2 dx + \int_0^1 f(M) z^+ dx \ge -\delta\int_0^1 \left|z^+\right|^2 dx = -\delta\left\|z^+(t)\right\|^2. \end{aligned} \tag{4.9}$$

Hence, it follows from (4.6), (4.8), (4.9) that

$$\frac{1}{2}\frac{d}{dt}\left\|z^+(t)\right\|^2 + a_0\left\|z^+(t)\right\|_{H^1}^2 \le \delta\left\|z^+(t)\right\|^2. \tag{4.10}$$

Integrating (4.10), we get

$$\left\|z^+(t)\right\|^2 \le \left\|z^+(0)\right\|^2 + 2\delta\int_0^t \left\|z^+(s)\right\|^2 ds. \tag{4.11}$$

Since $z^+(0) = (u(x,0)-M)^+ = (u_0(x)-M)^+ = 0$, hence, using Gronwall's Lemma, we obtain $\left\|z^+(t)\right\|^2 = 0$. Thus $z^+ = 0$ and $u(x,t) \le M$, for *a.e.* $(x,t) \in Q_T$.

The case $-M \le u_0(x)$, *a.e.*, $x \in \Omega$, and $M \ge \max\left\{\frac{1}{h_0}\left\|g_0\right\|_{L^\infty(0,T)}, \frac{1}{h_1}\left\|g_1\right\|_{L^\infty(0,T)}\right\}$ can be dealt with, in the same manner as above, by considering $z = u + M$ and $z^- = \frac{1}{2}(|z| - z)$, we also obtain $z^- = 0$ and hence $u(x,t) \ge -M$, for *a.e.* $(x,t) \in Q_T$.

From all above, one obtains $|u(x,t)| \le M$, *a.e.* $(x,t) \in Q_T$, i.e.,

$$\|u\|_{L^\infty(Q_T)} \le M, \tag{4.12}$$

for all $M \ge \max\left\{\left\|u_0\right\|_{L^\infty}, \frac{1}{h_0}\left\|g_0\right\|_{L^\infty(0,T)}, \frac{1}{h_1}\left\|g_1\right\|_{L^\infty(0,T)}\right\}$.

This implies (4.1). Theorem 4.1 is proved.■

5 Asymptotic behavior of the solution as $t \to +\infty$.

In this part, let $T > 0$, $(H_1) - (H_7)$ hold. Then, there exists a unique solution u of problem (1.1) – (1.3) such that

$$\begin{cases} u \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2) \cap L^p(Q_T), \\ \\ tu \in L^\infty(0,T;H^1), \;\; tu' \in L^2(Q_T). \end{cases}$$

We shall study asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We make the following supplementary assumptions on the functions $\mu\,(x,t)$, $f_1\,(x,t)$, $g_1(t)$, $g_2(t)$.

(H_3'') $g_0,\, g_1 \in W^{1,1}(\mathbb{R}_+)$,

(H_4'') $\mu \in C^1([0,1] \times \mathbb{R}_+)$, $\mu(x,t) \ge \mu_0 > 0$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$,

(H_5'') $f_1 \in L^\infty(0,\infty;L^2)$,

(H_6'') There exist the positive constants C_1, γ_1, $g_{0\infty}$, $g_{1\infty}$ and the functions

$\mu_\infty \in C^1([0,1])$, $f_{1\infty} \in L^2$, such that

(i) $|g_0(t) - g_{0\infty}| \le C_1 e^{-\gamma_1 t}$, $\forall t \ge 0$,

(ii) $|g_1(t) - g_{1\infty}| \le C_1 e^{-\gamma_1 t}$, $\forall t \ge 0$,

(iii) $\left\|\mu\,(t) - \mu_\infty\right\|_{L^\infty} \le C_1 e^{-\gamma_1 t}$, $\forall t \ge 0$, $\mu_\infty(x) \ge \mu_0 > 0, \forall x \in [0,1]$,

(iv) $\left\|f_1(t) - f_{1\infty}\right\| \le C_1 e^{-\gamma_1 t}$, $\forall t \ge 0$.

First, we consider the following stationary problem

$$\begin{cases} -\frac{\partial}{\partial x}\left[\mu_\infty(x)\, u_x\right] + f(u) = f_{1\infty}(x),\ 0 < x < 1, \\ u_x(0) = h_0 u(0) + g_{0\infty},\ \ -u_x(1) = h_1 u(1) + g_{1\infty}. \end{cases} \tag{5.1}$$

The weak solution of problem (5.1) is obtained from the following variational problem.

Find $u_\infty \in H^1$ such that

$$a_\infty(u_\infty, v) + \langle f(u_\infty), v\rangle = \langle f_{1\infty}, v\rangle - \mu_\infty(0) g_{0\infty} v(0) - \mu_\infty(1) g_{1\infty} v(1), \tag{5.2}$$

for all $v \in H^1$, where

$$\begin{aligned} a_\infty(u, v) &= \int_0^1 \mu_\infty(x) u_x(x) v_x(x) dx + h_0 \mu_\infty(0) u(0) v(0) + h_1 \mu_\infty(1) u(1) v(1) \\ &= \langle \mu_\infty u_x, v_x\rangle + h_0 \mu_\infty(0) u(0) v(0) + h_1 \mu_\infty(1) u(1) v(1), \text{ for all } u, v \in H^1. \end{aligned} \tag{5.3}$$

We then have the following theorem.

Theorem 5.1. *Let* (H_6), $(H_3'') - (H_6'')$ *hold. Then there exists a solution* u_∞ *of the variational problem* (5.2) *such that* $u_\infty \in H^1$.

Furthermore, if f *satisfies the following condition, in addition,*

(H_7'') $\quad f(u) + \delta u$ *is nondecreasing with respect to variable* u, *with* $0 < \delta < a_0$.

Then the solution is unique.

Proof. Denote by $\{w_j\}$, $j = 1, 2, ...$an orthonormal basis in the separable Hilbert space H^1. Put

$$y_m = \sum_{j=1}^{m} d_{mj} w_j, \tag{5.4}$$

where d_{mj} satisfy the following nonlinear equation system:

$$a_\infty(y_m, w_j) + \langle f(y_m), w_j\rangle = \langle f_{1\infty}, w_j\rangle - \mu_\infty(0) g_{0\infty} w_j(0) - \mu_\infty(1) g_{1\infty} w_j(1),\ 1 \le j \le m. \tag{5.5}$$

By the Brouwer's lemma (see Lions [5], Lemma 4.3, p. 53), it follows from the hypotheses (H_6), $(H_3'') - (H_6'')$ that system (5.4), (5.5) has a solution y_m.

Multiplying the j^{th} equation of system (5.5) by d_{mj}, then summing up with respect to j, we have

$$a_\infty(y_m, y_m) + \langle f(y_m), y_m\rangle = \langle f_{1\infty}, y_m\rangle - \mu_\infty(0) g_{0\infty} y_m(0) - \mu_\infty(1) g_{1\infty} y_m(1). \tag{5.6}$$

By using the inequality (2.3) and by the hypotheses (H_6), $(H_3'') - (H_6'')$, we obtain

$$a_0 \|y_m\|_{H^1}^2 + C_1 \|y_m\|_{L^p}^p \le C_1' + \left[\|f_{1\infty}\| + \sqrt{2}\left(|\mu_\infty(0) g_{0\infty}| + |\mu_\infty(1) g_{1\infty}|\right)\right] \|y_m\|_{H^1}. \tag{5.7}$$

Hence, we deduce from (5.7) that

$$\begin{cases} \|y_m\|_{H^1} \le C, \\ \|y_m\|_{L^p} \le C, \end{cases} \tag{5.8}$$

C is a constant independent of m.

By means of (5.8) and Lemma 2.1, the sequence $\{y_m\}$ has a subsequence still denoted by $\{y_m\}$ such that

$$\begin{cases} y_m \to u_\infty & \text{in } \ H^1 \ \text{ weakly,} \\ y_m \to u_\infty & \text{in } \ L^2 \text{ strongly and } \ a.e. \text{ in } \ \Omega, \\ y_m \to u_\infty & \text{in } \ L^p \ \text{ weakly.} \end{cases} \tag{5.9}$$

On the other hand, by $(5.9)_2$ and (H_6), we have

$$f(y_m) \to f(u_\infty) \ \text{ a.e. in } \ \Omega. \tag{5.10}$$

We also deduce from the hypothesis (H_6) and from $(5.8)_2$ that

$$\textstyle\int_0^1 |f(y_m(x))|^{p'} dx \leq 2^{p'-1} C_2^{p'} [1 + \int_0^1 |y_m(x)|^p dx] \leq C, \tag{5.11}$$

where C is a constant independent of m.

Applying Lemma 3.4 with $N = 1$, $q = p'$, $G_m = f(y_m)$, $G = f(u_\infty)$, we deduce from (5.10), (5.11) that

$$f(y_m) \to f(u_\infty) \text{ in } L^{p'} \ \text{ weakly.} \tag{5.12}$$

Passing to the limit in Eq. (5.5), we find without difficulty from (5.9), (5.12) that u_∞ satisfies the equation

$$a_\infty(u_\infty, w_j) + \langle f(u_\infty), w_j\rangle = \langle f_{1\infty}, w_j\rangle - \mu_\infty(0) g_{0\infty} w_j(0) - \mu_\infty(1) g_{1\infty} w_j(1). \tag{5.13}$$

Equation (5.13) holds for every $j = 1, 2, ...$, i.e., (5.2) holds.

The solution of the problem (5.2) is unique; that can be showed using the same arguments as in the proof of Theorem 3.1.■

Now we consider asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We then have the following theorem.

Theorem 5.2. *Let* (H_1), (H_2), (H_6), $(H_3'') - (H_6'')$, (H_7'') *hold. Then we have*

$$\|u(t) - u_\infty\|^2 \leq \left(\|u_0 - u_\infty\|^2 + \tfrac{4C}{\varepsilon(\gamma_1 - \gamma)}\right) e^{-2\gamma t}, \ \forall t \geq 0, \tag{5.14}$$

where

$$0 < \gamma < \min\{\gamma_1, \, a_0 - \delta - 4\varepsilon\}, \ \ 0 < 4\varepsilon < a_0 - \delta,$$

$C > 0$ *is a* a constant independing of t.

Proof. Put $Z_m(t) = u_m(t) - y_m$. Let us subtract $(3.5)_1$ with (5.5) to obtain

$$\begin{cases} \langle Z_m'(t), w_j\rangle + a(t; u_m(t), w_j) - a_\infty(y_m, w_j) + \langle f(u_m(t)) - f(y_m), w_j\rangle \\ \qquad = \langle f_1(t) - f_{1\infty}, w_j\rangle - [\mu(0,t) g_0(t) - \mu_\infty(0) g_{0\infty}] w_j(0) \\ \qquad\quad - [\mu(1,t) g_1(t) - \mu_\infty(1) g_{1\infty}] w_j(1), \ 1 \leq j \leq m, \\ Z_m(0) = u_{0m} - y_m. \end{cases} \tag{5.15}$$

By multiplying $(5.15)_1$ by $c_{mj}(t) - d_{mj}$ and summing up in j, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left\|Z_m(t)\right\|^2 + a(t; Z_m(t), Z_m(t)) + a(t; y_m, Z_m(t)) - a_\infty(y_m, Z_m(t)) \\
&\qquad\qquad + \langle f(u_m(t)) - f(y_m), Z_m(t)\rangle \\
&\qquad = \langle f_1(t) - f_{1\infty}, Z_m(t)\rangle - \left[\mu\left(0,t\right) g_0(t) - \mu_\infty(0) g_{0\infty}\ \right] Z_m(0,t) \\
&\qquad\quad - \left[\mu\left(1,t\right) g_1(t) - \mu_\infty(1) g_{1\infty}\ \right] Z_m(1,t).
\end{aligned} \tag{5.16}$$

By the assumptions $(H_3'') - (H_6'')$, (H_7''), and using the inequalities (2.2), (2.3), and with $\varepsilon > 0$, we estimate without difficulty the following terms in (5.16) as follows

$$a(t; Z_m(t), Z_m(t)) \geq a_0 \left\|Z_m(t)\right\|_{H^1}^2; \tag{5.17}$$

$$\langle f(u_m(t)) - f(y_m), Z_m(t)\rangle \geq -\delta \left\|Z_m(t)\right\|^2 \geq -\delta \left\|Z_m(t)\right\|_{H^1}^2; \tag{5.18}$$

$$\begin{aligned}
a(t; y_m, Z_m(t)) - a_\infty(y_m, Z_m(t)) &= \langle\left(\mu(t) - \mu_\infty\right) y_{mx}, Z_{mx}(t)\rangle \\
&\quad + h_0\left(\mu\left(0,t\right) - \mu_\infty(0)\right) y_m(0) Z_m(0,t) \\
&\quad + h_1\left(\mu\left(1,t\right) - \mu_\infty(1)\right) y_m(1) Z_m(1,t);
\end{aligned} \tag{5.19}$$

Note that $\left\|y_m\right\|_{H^1} \leq C$, we obtain from (5.19) that

$$\begin{aligned}
\left|a(t; y_m, Z_m(t)) - a_\infty(y_m, Z_m(t))\right| &\leq \left\|\mu\left(t\right) - \mu_\infty\right\|_{L^\infty} \left\|y_{mx}\right\| \left\|Z_{mx}(t)\right\| \\
&\quad + 2h_0 \left\|\mu\left(t\right) - \mu_\infty\right\|_{L^\infty} \left\|y_m\right\|_{H^1} \left\|Z_m(t)\right\|_{H^1} \\
&\quad + 2h_1 \left\|\mu\left(t\right) - \mu_\infty\right\|_{L^\infty} \left\|y_m\right\|_{H^1} \left\|Z_m(t)\right\|_{H^1} \\
&\leq \left(1 + 2h_0 + 2h_1\right) C_1 e^{-\gamma_1 t} C \left\|Z_m(t)\right\|_{H^1} \leq \varepsilon \left\|Z_m(t)\right\|_{H^1}^2 + \tfrac{1}{\varepsilon} C e^{-2\gamma_1 t};
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\left|\langle f_1(t) - f_{1\infty}, Z_m(t)\rangle\right| &\leq \left\|f_1(t) - f_{1\infty}\right\| \left\|Z_m(t)\right\| \\
&\leq C_1 e^{-\gamma_1 t} \left\|Z_m(t)\right\|_{H^1} \leq \varepsilon \left\|Z_m(t)\right\|_{H^1}^2 + \tfrac{1}{\varepsilon} C e^{-2\gamma_1 t};
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
&-\left[\mu\left(0,t\right) g_0(t) - \mu_\infty(0) g_{0\infty}\ \right] Z_m(0,t) \\
&\quad = -\left[\left(\mu\left(0,t\right) - \mu_\infty(0)\right) g_0(t) + \mu_\infty(0)\left(g_0(t) - g_{0\infty}\right)\ \right] Z_m(0,t) \\
&\quad \leq \sqrt{2} \left\|Z_m(t)\right\|_{H^1} \left[\left\|\mu\left(t\right) - \mu_\infty\right\|_{L^\infty} \left\|g_0\right\|_{L^\infty(\mathbb{R}_+)} + \mu_\infty(0) \left|g_0(t) - g_{0\infty}\right|\right] \\
&\quad \leq \sqrt{2} \left\|Z_m(t)\right\|_{H^1} \left[\left\|g_0\right\|_{L^\infty(\mathbb{R}_+)} + \mu_\infty(0)\right] C_1 e^{-\gamma_1 t} \leq \varepsilon \left\|Z_m(t)\right\|_{H^1}^2 + \tfrac{1}{\varepsilon} C e^{-2\gamma_1 t}.
\end{aligned} \tag{5.22}$$

Similarly

$$-\left[\mu\left(1,t\right) g_1(t) - \mu_\infty(1) g_{1\infty}\ \right] Z_m(1,t) \leq \varepsilon \left\|Z_m(t)\right\|_{H^1}^2 + \tfrac{1}{\varepsilon} C e^{-2\gamma_1 t}. \tag{5.23}$$

It follows from (5.16) – (5.18), (5.20) – (5.23) and (2.3), that

$$\tfrac{d}{dt}\left\|Z_m(t)\right\|^2 + 2\left(a_0 - \delta - 4\varepsilon\right)\left\|Z_m(t)\right\|_{H^1}^2 \leq \tfrac{8}{\varepsilon} C e^{-2\gamma_1 t}. \tag{5.24}$$

Choose $\varepsilon > 0$ and $\gamma > 0$ such that $a_0 - \delta - 4\varepsilon > 0$ and $\gamma < \min\{\gamma_1, a_0 - \delta - 4\varepsilon\}$, then we have from (5.24) that

$$\tfrac{d}{dt}\left\|Z_m(t)\right\|^2 + 2\gamma\left\|Z_m(t)\right\|^2 \leq \tfrac{8}{\varepsilon} C e^{-2\gamma_1 t}. \tag{5.25}$$

Hence, we obtain from (5.25) that

$$\left\|Z_m(t)\right\|^2 \leq \left[\left\|Z_m(0)\right\|^2 + \tfrac{4C}{\varepsilon(\gamma_1 - \gamma)}\right] e^{-2\gamma t}. \tag{5.26}$$

Letting $m \to +\infty$ in (5.26) we obtain

$$\left\|u(t) - u_\infty\right\|^2 \leq \liminf_{m\to+\infty}\left\|u_m(t) - y_m\right\|^2 \leq \left(\left\|u_0 - u_\infty\right\|^2 + \tfrac{4C}{\varepsilon(\gamma_1 - \gamma)}\right) e^{-2\gamma t}, \text{ for all } t \geq 0. \tag{5.27}$$

This completes the proof of Theorem 5.2.■

6 Numerical results

First, we present some results of numerical comparison of the approximated representation of the solution of a nonlinear problem of the type (1.1) – (1.3) and the corresponding exact solution of this problem.

Let the problem

$$\begin{cases} u_t - u_{xx} + f(u) = f_1(x,t), \ 0 < x < 1, \ t > 0, \\ u_x(0,t) = 2u(0,t) + g_0(t), \ \ -u_x(1,t) = u(1,t) + g_1(t), \\ u(x,0) = \widetilde{u}_0(x), \end{cases} \tag{6.1}$$

where

$$\begin{cases} f_1(x,t) = -e^x(1 + 2e^{-t}) + (1 + e^{-t})^{p-1} e^{(p-1)x}, \\ f(u) = |u|^{p-2} u, \ \ p = \tfrac{5}{2}, \\ g_0(t) = -1 - e^{-t}, \ \ g_1(t) = -2e(1 + e^{-t}), \\ \widetilde{u}_0(x) = 2e^x. \end{cases} \tag{6.2}$$

The exact solution of the problem (6.1), (6.2) is $u(x,t) = (1 + e^{-t})e^x$.

To solve numerically the problem (6.1), (6.2), we consider the nonlinear differential system for the unknowns $u_k(t) = u(x_k, t)$, $x_k = kh$, $h = 1/N$.

$$\begin{cases} \frac{du_k}{dt}(t) = \frac{1}{h^2}u_{k-1} - \frac{2}{h^2}u_k + \frac{1}{h^2}u_{k+1} - f(u_k) + f_1(x_k, t), \\ u_0 = \frac{1}{1+2h}\left(u_1 - hg_0(t)\right), \ u_N = \frac{1}{1+h}\left(u_{N-1} - hg_1(t)\right), \\ u_k(0) = \widetilde{u}_0(x_k), \ \ k = 1, 2, ..., N-1. \end{cases}$$

or

$$\begin{cases} \frac{du_1}{dt}(t) = \frac{-1}{h^2}\left(\frac{1+4h}{1+2h}\right)u_1 + \frac{1}{h^2}u_2 - f(u_1) - \frac{1}{h(1+2h)}g_0(t) + f_1(x_1,t), \\ \frac{du_k}{dt}(t) = \frac{1}{h^2}u_{k-1} - \frac{2}{h^2}u_k + \frac{1}{h^2}u_{k+1} - f(u_k) + f_1(x_k,t), \quad k = \overline{2, N-2}, \\ \frac{du_{N-1}}{dt}(t) = \frac{1}{h^2}u_{N-2} - \frac{1}{h^2}\left(\frac{1+2h}{1+h}\right)u_{N-1} - f(u_{N-1}) - \frac{1}{h(1+h)}g_1(t) + f_1(x_{N-1},t), \\ u_k(0) = \widetilde{u}_0(x_k), \quad k = \overline{1, N-1}. \end{cases} \tag{6.3}$$

To solve the nonlinear differential (6.3) at the time t, we use the following linear recursive scheme generated by the nonlinear term $f(u_k)$:

$$\begin{cases} \frac{du_1^{(n)}}{dt}(t) = \frac{-1}{h^2}\left(\frac{1+4h}{1+2h}\right)u_1^{(n)} + \frac{1}{h^2}u_2^{(n)} - f(u_1^{(n-1)}) - \frac{1}{h(1+2h)}g_0(t) + f_1(x_1,t), \\ \frac{du_k^{(n)}}{dt}(t) = \frac{1}{h^2}u_{k-1}^{(n)} - \frac{2}{h^2}u_k^{(n)} + \frac{1}{h^2}u_{k+1}^{(n)} - f(u_k^{(n-1)}) + f_1(x_k,t), \quad k = \overline{2, N-2}, \\ \frac{du_{N-1}^{(n)}}{dt}(t) = \frac{1}{h^2}u_{N-2}^{(n)} - \frac{1}{h^2}\left(\frac{1+2h}{1+h}\right)u_{N-1}^{(n)} - f(u_{N-1}^{(n-1)}) - \frac{1}{h(1+h)}g_1(t) + f_1(x_{N-1},t), \\ u_k^{(n)}(0) = \widetilde{u}_0(x_k), \quad k = \overline{1, N-1}. \end{cases} \tag{6.4}$$

The linear differential system (6.4) is solved by searching the associated eigenvalues and eigenfunctions. With a spatial step $h = \frac{1}{5}$ on the interval $[0,1]$ and for $t \in [0,3]$, we have drawn the corresponding approximate surface solution $(x,t) \longrightarrow u(x,t)$ in figure 1, obtained by successive re-initializations in t with a time step $\Delta t = \frac{1}{50}$. For comparison in figure 2, we have also drawn the exact surface solution $(x,t) \longrightarrow u(x,t)$.

Note that, the approximate solution $u(x,t)$ decreases exponentially to $u_\infty(x)$ as t tends to infinity, u_∞ being the unique solution of the corresponding steady state problem

$$\begin{cases} -u_{xx} + |u|^{\frac{1}{2}}\, u = -e^x + e^{\frac{3}{2}x}, \ 0 < x < 1, \\ u_x(0) = 2u(0) - 1, \ -u_x(1) = u(1) - 2e. \end{cases} \tag{6.5}$$

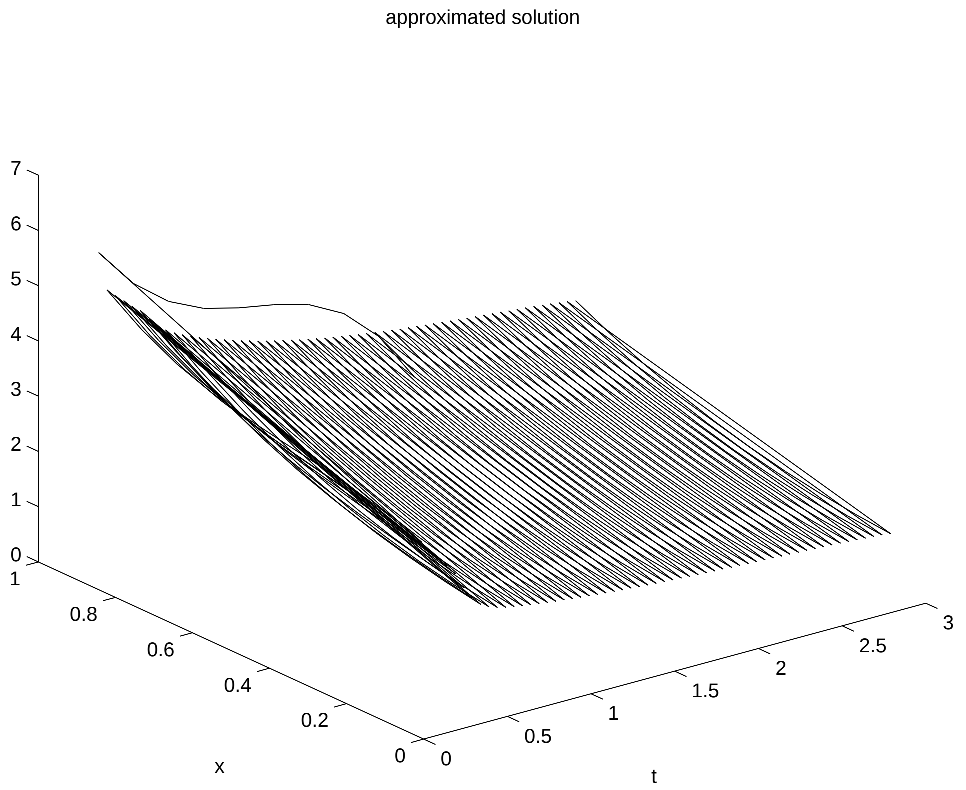

Figure 1. Approximated solution

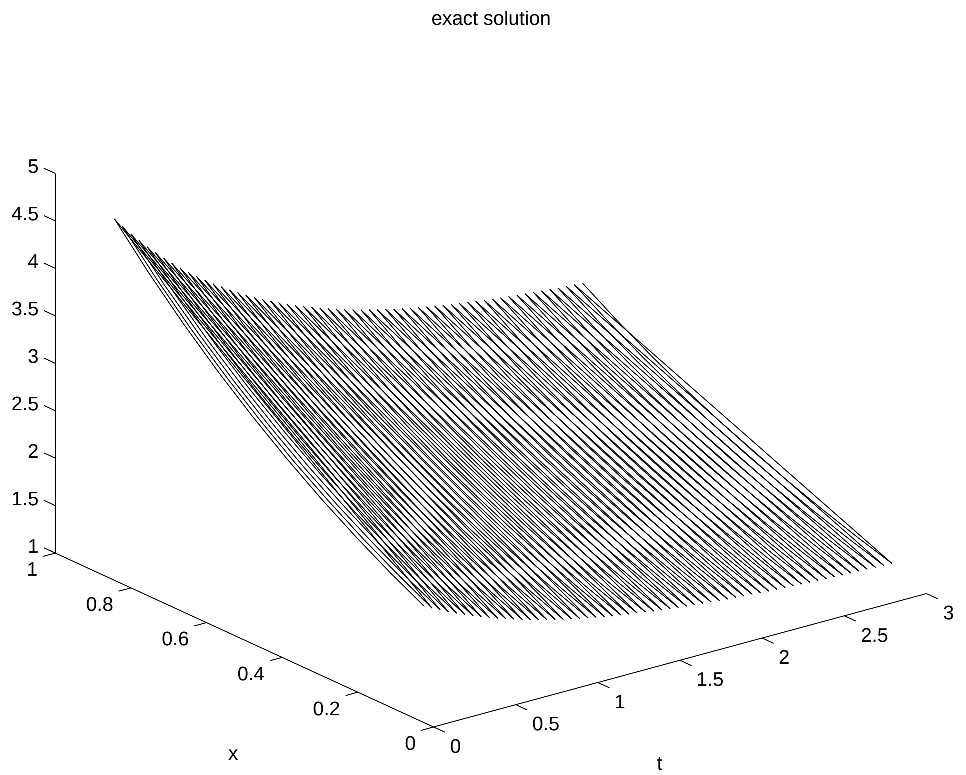

Figure 2. Exact solution

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